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Topological gauge field mass generation by toroidal spacetime

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Abstract. We consider an Abelian gauge field theory on Euclidean partially compactified spacetime $T^N \times \mathbb{R}^n$ for arbitrary N and n. The one-loop effective potential generated by quantum fluctuations of a massive scalar field, minimally coupled to a constant background Abelian gauge potential is calculated for arbitrary compactification lengths $L_1, \ldots, L_N$ of the multi-dimensional torus. In particular the topologically generated mass of the gauge field is obtained and its complicated dependence on the parameters involved (compactification lengths, mass M of the scalar field) is given explicitly. It is found that the topologically generated mass is positive for arbitrary N and n and that it does not depend on a renormalization parameter. For $n > 2$ the limit $M \to 0$ is smooth, but for $n = 0, 1, 2$ zero modes play a crucial role and the generation of real or imaginary gauge field masses is possible.

1. Introduction

Topological mass generation and the Casimir effect are beautiful and simple manifestations of the influence that boundaries or non-trivial spacetime topologies have on quantum field theories (see for example [1–19]). The vacuum structure depends strongly on the global structure of spacetime and in general it is difficult to obtain even the sign of resulting Casimir forces, or to say that topologically generated masses are real or imaginary (see however [8, 20]).

Explicit results may only be derived for highly symmetric configurations [2–4, 8–11, 20–24]. The most familiar example of topological mass generation is Euclidean finite-temperature field theory, in which all fields are defined on the spacetime cylinder $S^1 \times \mathbb{R}^n$, $n \in \mathbb{N}_0$. The resulting topological mass of the time component of the gauge potential is known as the inverse plasma screening length of the finite-temperature gauge theory.

In the generalization to finite-temperature quantum field theory we consider a Euclidean, Abelian gauge theory in $T^N \times \mathbb{R}^n$, $N \in \mathbb{N}$, where the different toroidal components are assumed to have arbitrary compactification lengths $L_1, \ldots, L_N$. These considerations are mainly based on the work of Actor [25–27], where a lot of literature relevant in the context of topological mass generation may also be found.

A massive complex scalar field ϕ defined on $T^N \times \mathbb{R}^n$ with periodic boundary conditions for each of the toroidal components is coupled to a constant gauge potential $A_\mu = (A_1, \ldots, A_N, \mathbf{A})$, $\mathbf{A} \in \mathbb{R}^n$. Due to the non-trivial topology, constant values of the toroidal components A_a, $a = 1, \ldots, N$, are physical parameters of the theory and the effective potential of the gauge theory will depend on these parameters.

We will give a more detailed analysis of the theory in section 2, then calculate the one-loop potential in sections 3 and 4. The main technical complication arises because of the

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arbitrary compactification lengths of the toroidal components. Two different derivations are given, one especially useful for large values of the mass of the quantum field ϕ (section 3) and, in order to analyze the massless quantum field (see section 5), one for small values (section 4). The results involve different types of generalized Epstein zeta functions and the relevant properties of them are summarized. Given the exact one-loop effective potential it is possible to read off the topologically generated masses of the toroidal components of the gauge potential and to discuss some of their properties.

2. Abelian gauge field theory on $T^N \times \mathbb{R}^n$

Let $z = (x, y)$ be a position vector in $T^N \times \mathbb{R}^n$, where $x \in T^N$, $y \in \mathbb{R}^n$. Then the Abelian, Euclidean gauge field theory we consider may be described by the field equation

$$\left\{ -\sum_{i=1}^{N} \left(\frac{\partial}{\partial x_i} - \mathrm{i}A_i \right)^2 - \Delta + M^2 \right\} \phi(x, y) = 0 \tag{2.1}$$

with periodic boundary conditions for $\phi(x, y)$ in the toroidal directions, $\phi(x_i, y) = \phi(x_i + L_i, y)$. Here, Δ is the Laplacian of $\mathbb{R}^n$ and the A_i's are the constant toroidal components of the gauge potential (a constant part $A \in \mathbb{R}^n$ is a non-physical parameter and has been gauged away), which are angular variables of the theory, $-\pi \leqslant L_i A_i \leqslant \pi$ [24]. Furthermore, to ensure gauge invariance of the theory, the constant values of A have to be interpreted as $\oint A$ [28].

We are interested in the one-loop effective potential $V(A_1, \ldots, A_N; L_1, \ldots, L_N)$ of the theory (as an abbreviation we write $V(A_i; L_i)$). In the zeta function regularization scheme it is defined by [29, 30]

$$V_T V_{\mathrm{eff}}(A_i; L_i) = \zeta(0; A_i; L_i) \ln \lambda^2 - \zeta'(0; A_i; L_i) \tag{2.2}$$

where λ is a scaling length, $V_T = L_1 \times \cdots \times L_N$ is the volume of the torus and the prime denotes differentiation with respect to the first argument (this means with respect to s, see equation (2.3)). $\zeta(s; A_i; L_i)$ is the zeta function associated with the operator (2.1) this means for $\mathrm{Re}\, s > \frac{1}{2}(n + N)$

$$\zeta(s; u_i; w_i) = \left(\frac{\sqrt{\pi}}{L_1} \right)^n \frac{\Gamma\left(s - \frac{1}{2}n\right)}{\Gamma(s)} \left(\frac{L_1}{2\pi} \right)^{2s} Z_N^{v^2}\left(s - \tfrac{1}{2}n; w_1, \ldots, w_N; u_1, \ldots, u_N\right) \tag{2.3}$$

where we have introduced the dimensionless parameters $v^2 = (L_1 M / 2\pi)^2$, $u_i = A_i L_i / 2\pi$, $w_i = (L_1 / L_i)^2$, and the generalized Epstein zeta function

$$Z_N^{v^2}(\nu; w_1, \ldots, w_N; u_1, \ldots, u_N) = \sum_{l_1, \ldots, l_N = -\infty}^{\infty} [w_1 (l_1 - u_1)^2 + \cdots + w_N (l_N - u_N)^2 + v^2]^{-\nu} \tag{2.4}$$

valid for $\mathrm{Re}\, s > N/2$.

To find the effective potential (2.2) we need the derivative of equation (2.3) at $s = 0$.

Let us first express equation (2.2) in terms of properties of the generalized Epstein zeta function $Z_N^{v^2}(s; w_1, \ldots, w_N; u_1, \ldots, u_N)$.

Using regularization techniques for Mellin transforms [31,32], it is easy to show, that for N even the poles of order one of $Z_N^{v^2}(s; w_1, \ldots, w_N; u_1, \ldots, u_N)$ are located at $s = \frac{1}{2}N; \frac{1}{2}N-1; \ldots; 1$, whereas for N odd one finds $s = \frac{1}{2}N; \frac{1}{2}N-1; \ldots; \frac{1}{2}; -\frac{1}{2}(2l+1)$, $l \in \mathbb{N}_0$. The residuum

$$\text{Res}\, Z_N^{v^2}(j; w_1, \ldots, w_N; u_1, \ldots, u_N) = \frac{(-1)^{\frac{1}{2}N+j}\pi^{N/2}}{\sqrt{w_1 \cdots w_N}} \frac{v^{N-2j}}{\Gamma(j)\left(\frac{1}{2}N-j\right)!} \tag{2.5}$$

does not depend on the toroidal components A_j of the gauge potential.

In addition for $p \in \mathbb{N}_0$ one has

$$Z_N^{v^2}(-p; w_1, \ldots, w_N; u_1, \ldots, u_N) = \begin{cases} 0 & \text{for } N \text{ odd} \\ \dfrac{(-1)^{N/2} p! \pi^{N/2}}{\sqrt{w_1 \cdots w_N}} \dfrac{v^{N+2p}}{\left(\frac{1}{2}N+p\right)!} & \text{for } N \text{ even.} \end{cases} \tag{2.6}$$

Due to the different pole structure for N even and N odd, and, furthermore, because of the different behaviour of $\Gamma\left(s-\frac{1}{2}n\right)/\Gamma(s)$ at $s = 0$ for n even and n odd, one has to consider four different situations. Introducing $PP\ Z_N^{v^2}$ for the finite part of $Z_N^{v^2}$, the different results for the effective potential read:

(i) $n = 2k$, $k \in \mathbb{N}_0$, N even

$$\begin{aligned} V_T V_{\text{eff}}(w_i; u_i) = -\left(\frac{\sqrt{\pi}}{L_1}\right)^n \frac{(-1)^k}{k!} \{ & Z_N'^{\,v^2}(-k; w_1, \ldots, w_N; u_1, \ldots, u_N) \\ & + Z_N^{v^2}(-k; w_1, \ldots, w_N; u_1, \ldots, u_N)\,[2\ln(L_1/2\pi\lambda) + \gamma + \psi(k+1)]\} \end{aligned} \tag{2.7}$$

(ii) $n = 2k$, $k \in \mathbb{N}_0$, N odd

$$V_T V_{\text{eff}}(w_i; u_i) = -\left(\frac{\sqrt{\pi}}{L_1}\right)^n \frac{(-1)^k}{k!} Z_N'^{\,v^2}(-k; w_1, \ldots, w_N; u_1, \ldots, u_N) \tag{2.8}$$

(iii) $n = 2k+1$, $k \in \mathbb{N}_0$, N even

$$V_T V_{\text{eff}}(w_i; u_i) = -\left(\frac{\sqrt{\pi}}{L_1}\right)^n \Gamma(-k-\tfrac{1}{2}) Z_N^{v^2}\left(-k-\tfrac{1}{2}; w_1, \ldots, w_N; u_1, \ldots, u_N\right) \tag{2.9}$$

(iv) $n = 2k+1$, $k \in \mathbb{N}_0$, N odd

$$\begin{aligned} V_T V_{\text{eff}}(w_i; u_i) &= -\left(\frac{\sqrt{\pi}}{L_1}\right)^n \Gamma(-k-\tfrac{1}{2}) \Big\{ PP\ Z_N^{v^2}\left(-k-\tfrac{1}{2}; w_1, \ldots, w_N; u_1, \ldots, u_N\right) \\ &\quad + \text{Res}\, Z_N^{v^2}\left(-k-\tfrac{1}{2}; w_1, \ldots, w_N; u_1, \ldots, u_N\right) \\ &\quad \times \left[2\ln\left(\frac{L_1}{2\pi\lambda}\right) + \gamma + \psi\left(k+\frac{3}{2}\right)\right]\Big\} \end{aligned} \tag{2.10}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ and $\gamma = -\psi(1)$.

In order to obtain the effective potential, the remaining task is to construct analytical continuations of $Z_N^{v^2}(\nu; w_1, \ldots, w_N; u_1, \ldots, u_N)$, equation (2.4) to $\text{Re}\,\nu < N/2$ and to determine the properties of $Z_N^{v^2}(\nu; w_1, \ldots, w_N; u_1, \ldots, u_N)$ needed in equations (2.7)–(2.10). This will be done in the following sections, where different approaches useful for large and small values of v^2 are described.

3. Calculation of the effective potential for large values of v^2

Let us first derive a representation of the effective potential which is useful for large values of v^2.

As mentioned in section 2 we need the analytical continuation of the function $Z_N^{v^2}(s; w_1, \ldots, w_N; u_1, \ldots, u_N)$, equation (2.4), to $\mathrm{Re}\, s < N/2$. Writing, as usual, equation (2.4) in the form of a Mellin transform and performing re-summations employing for $t \in \mathbb{R}_+$, $z \in \mathbb{C}$ [33]

$$\sum_{n=-\infty}^{\infty} \exp\{-tn^2 + 2\pi i n z\} = \left(\frac{\pi}{t}\right)^{1/2} \sum_{n=-\infty}^{\infty} \exp\left\{-\frac{\pi^2}{t}(n-z)^2\right\} \tag{3.1}$$

which is due to Jacobi's relation between theta functions, one finds

$$\begin{aligned} Z_N^{v^2}(s; w_1, \ldots, w_N; u_1, \ldots, u_N) &= \frac{\pi^{N/2}}{\sqrt{w_1 \cdots w_N}} \frac{\Gamma(s - \frac{1}{2}N)}{\Gamma(s)} v^{N-2s} \\ &\quad + \frac{\pi^s}{\sqrt{w_1 \cdots w_N}} \frac{2}{\Gamma(s)} \sum_{l_1, \ldots, l_N = -\infty}^{\infty}{}' \exp\{2\pi i [l_1 u_1 + \cdots + l_N u_N]\} \\ &\quad \times v^{\frac{1}{2}N - s} \left[\frac{l_1^2}{w_1} + \cdots + \frac{l_N^2}{w_N}\right]^{\frac{1}{2}(s - \frac{1}{2}N)} K_{\frac{1}{2}N - s}\left(2\pi v \left[\frac{l_1^2}{w_1} + \cdots + \frac{l_N^2}{w_N}\right]^{1/2}\right) \end{aligned} \tag{3.2}$$

where the prime means omission of the summation index $l_1 = \cdots = l_N = 0$ (for similar considerations see [34]).

Using equation (3.2), the relevant information in equations (2.7)–(2.10) may be calculated and the effective potential is determined to be

$$\begin{aligned} V_{\mathrm{eff}}(w_i; u_i) &= -2 L_1^{-n-N} v^{\frac{1}{2}(N+n)} \sum_{l_1, \ldots, l_N = -\infty}^{\infty}{}' \exp\{2\pi i [l_1 u_1 + \cdots + l_N u_N]\} \\ &\quad \times \left[\frac{l_1^2}{w_1} + \cdots + \frac{l_N^2}{w_N}\right]^{-\frac{1}{4}(N+n)} K_{\frac{1}{2}(N+n)}\left(2\pi v \left[\frac{l_1^2}{w_1} + \cdots + \frac{l_N^2}{w_N}\right]^{1/2}\right) \\ &\quad - \left(\frac{\sqrt{\pi}}{L_1}\right)^{n+N} v^{N+n} \\ &\quad \times \begin{cases} \dfrac{(-1)^{\frac{1}{2}(n+N)}}{(\frac{1}{2}(n+N))!} [\psi(1 + \frac{1}{2}(n+N)) + \gamma - 2\ln(\lambda M)] & \text{for } n+N \text{ even} \\ \Gamma(-\frac{1}{2}(n+N)) & \text{for } n+N \text{ odd.} \end{cases} \end{aligned} \tag{3.3}$$

We are interested in the topologically generated masses m_{T_i} of the toroidal components A_i of the gauge potential. Given the expansion of $V_{\mathrm{eff}}(w_i; u_i)$ in powers of the A_i's, these masses $m_{T_i}^2$ are defined as the coefficients of the quadratic terms in that expansion

$$\begin{aligned} V_{\mathrm{eff}}(w_i; u_i) &= \sum C_{k_1 \cdots k_N} A_1^{k_1} \cdots A_N^{k_N} \\ &= \frac{1}{2g^2} \left[m_{T_1}^2 A_1^2 + \cdots + m_{T_N}^2 A_N^2\right] + \text{non-quadratic terms} \end{aligned} \tag{3.4}$$

where the gauge coupling g^2 has dimension $(\text{mass})^{4-N-n}$.

First, equation (3.3) shows that the masses $m_{T_j}^2$ are independent of the renormalization scale λ. This is already obvious from equations (2.7)–(2.10), because $Z_N^{\nu^2}(-k; w_1, \ldots, w_N; u_1, \ldots, u_N)$ (respectively Res $Z_N^{\nu^2}(-k-\frac{1}{2}; w_1, \ldots, w_N; u_1, \ldots, u_N)$) is independent of the gauge potential.

In order to determine the masses $m_{T_j}^2$ the full expansion of $V_{\text{eff}}(w_i; u_i)$ in powers of the A_i's is obtained. The relevant part of equation (3.3) reads

$$R(u_1, \ldots, u_N) = \sum_{l_1, \ldots, l_N=-\infty}^{\infty}{}' \exp\{2\pi i[l_1 u_1 + \cdots + l_N u_N]\} f(l_1, \ldots, l_N)$$

where $f(l_1, \ldots, l_N)$ contains only quadratic powers of the summation indices l_i, $i = 1, \ldots, N$. Using some multi-index notation, that is for $\nu(l) = (\nu_1, \ldots, \nu_l)$, $\nu_i \geqslant 0$, $i = 1, \ldots, l$, and $x = (x_1, \ldots, x_l)$, we define

$$\nu! := \nu_1! \cdots \nu_l! \qquad |\nu| := \sum_{i=1}^{l} \nu_i$$

and

$$x^{\nu(l)} = x_1^{\nu_1} \times \cdots \times x_l^{\nu_l}$$

the Taylor series expansion of $R(u_1, \ldots, u_N)$ reads

$$R(u_1, \ldots, u_N) = \sum_{j=0}^{\infty} (-1)^j (2\pi)^{2j} \sum_{t=1}^{N} 2^t \sum_{\{i_1, \ldots, i_t\}} \sum_{|\nu(t)|=j} \frac{1}{(2\nu(t))!} \times \sum_{l_{i_1}, \ldots, l_{i_t}=1}^{\infty} f(l_{i_1}, \ldots, l_{i_t}) x^{2\nu(t)}(i_1, \ldots, i_t) \tag{3.5}$$

where $x(i_1, \ldots, i_t) = (l_{i_1} u_{i_1}, \ldots, l_{i_t} u_{i_t})$ and $\sum_{\{i_1, \ldots, i_t\}}$ denotes the sum over all possible choices of the $i_1 < \cdots < i_t$ among $1, \ldots, N$.

As a result, equation (3.3) may be given in the form

$$\begin{aligned}
V_{\text{eff}}(w_i; u_i) = & -2L_1^{-n-N} v^{\frac{1}{2}(n+N)} \sum_{j=0}^{\infty} (-1)^j (2\pi)^{2j} \\
& \times \sum_{t=1}^{N} \sum_{\{i_1, \ldots, i_t\}} \sum_{|\nu(t)|=j} \frac{1}{(2\nu(t))!} \sum_{l_{i_1}, \ldots, l_{i_t}=1}^{\infty} 2^t \left[\frac{l_{i_1}^2}{w_{i_1}} + \cdots + \frac{l_{i_t}^2}{w_{i_t}} \right]^{-\frac{1}{4}(n+N)} \\
& \times K_{\frac{1}{2}(n+N)} \left(2\pi v \left[\frac{l_{i_1}^2}{w_{i_1}} + \cdots + \frac{l_{i_t}^2}{w_{i_t}} \right]^{1/2} \right) x^{2\nu(t)}(i_1, \ldots, i_t) \\
& - \left(\frac{\sqrt{\pi}}{L_1} \right)^{n+N} v^{n+N} \\
& \times \begin{cases} \dfrac{(-1)^{\frac{1}{2}(n+N)}}{(\frac{1}{2}(n+N))!} [\psi(1 + \frac{1}{2}(n+N)) + \gamma - 2\ln(\lambda M)] & \text{for } n+N \text{ even} \\ \Gamma(-\frac{1}{2}(n+N)) & \text{for } n+N \text{ odd.} \end{cases}
\end{aligned} \tag{3.6}$$

Contributions to the topological mass only arise for $|\nu(t)| = 1$ and one finds

$$m_{T_j}^2 = g^2 L_j^2 L_1^{-n-N} v^{\frac{1}{2}(n+N)} \sum_{m=1}^{N} \sum_{\{j_1,\ldots,j_{m-1}\}} \sum_{l_j,l_{j_1},\ldots,l_{j_{m-1}}=1}^{\infty} 2^{m+1} l_j^2 \times \left[\frac{l_j^2}{w_j} + \frac{l_{j_1}^2}{w_{j_1}} + \cdots + \frac{l_{j_{m-1}}^2}{w_{j_{m-1}}}\right]^{-\frac{1}{4}(n+N)} \times K_{\frac{1}{2}(n+N)}\left(2\pi v\left[\frac{l_j^2}{w_j} + \frac{l_{j_1}^2}{w_{j_1}} + \cdots + \frac{l_{j_{m-1}}^2}{w_{j_{m-1}}}\right]^{1/2}\right) \tag{3.7}$$

where now $\sum_{\{j_1,\ldots,j_{m-1}\}}$ denotes the sum over all possible choices of the indices $j_1 < \cdots < j_{m-1}$ among $1, \ldots, j-1, j+1, \ldots, N$.

For an equilateral torus $L_1 = \cdots = L_N = L$ the result reduces to

$$m_T^2 = g^2 L^2 \left(\frac{M}{2\pi L}\right)^{\frac{1}{2}(n+N)} \sum_{m=1}^{N} 2^{m+1} \binom{N-1}{m-1} \times \sum_{l_1,\ldots,l_m=1}^{\infty} l_1^2 [l_1^2 + \cdots + l_m^2]^{-\frac{1}{4}(n+N)} K_{\frac{1}{2}(n+N)}\left(ML[l_1^2 + \cdots + l_m^2]^{1/2}\right). \tag{3.8}$$

Obviously the masses $m_{T_j}^2$ are positive for arbitrary $n, N, M > 0$ and $L_1, \ldots, L_N$, and exponentially damped for $v \to \infty$, that is for fixed compactification lengths for $M \to \infty$. To give some idea of this strong dependence, in table 1 we give some numerical values of the topologically generated mass for the case $n+N = 4$ and for some values of the relevant product LM (restricting to the equilateral torus and $gM = \pi$). It is seen, that due to the exponentiell decay of the Kelvin function the value of the mass m_T^2 for increasing values of LM goes very quickly to zero.

Table 1. Values of the topologically generated mass for $LM = 0.1, 1, 10$.

LM	0.1	1	10
n=3, N=1	313	1.97	0.00002
n=2, N=2	1045	3.35	0.00002
n=1, N=3	5593	5.86	0.00002

To establish a connection between the topologically generated mass in $N-1$ toroidal, $n+1$ free dimensions and N toroidal, n free dimensions, let us change the notation for $m_{T_j}^2$ in equation (3.7) for a moment. We now write $m_{T_j}^2(N)$ to remind ourselves that it is the mass of the gauge potential A_j generated in N compactified dimensions. Then, for $j = 1, \ldots, N-1$, we find as a consequence of equation (3.7)

$$m_{T_j}^2(N) = m_{T_j}^2(N-1) + g^2 L_j^2 L_1^{-n-N} v^{\frac{1}{2}(n+N)} \times \sum_{m=2}^{N} \sum_{\{t_1,\ldots,t_{m-2}\}} \sum_{l_j,l_N,l_{t_1},\ldots,l_{t_{m-2}}=1}^{\infty} 2^{m+1} l_j^2 \times \left[\frac{l_j^2}{w_j} + \frac{l_N^2}{w_N} + \frac{l_{t_1}^2}{w_{t_1}} + \cdots + \frac{l_{t_{m-2}}^2}{w_{t_{m-2}}}\right]^{-\frac{1}{4}(n+N)} \times K_{\frac{1}{2}(n+N)}\left(2\pi v\left[\frac{l_j^2}{w_j} + \frac{l_N^2}{w_N} + \frac{l_{t_1}^2}{w_{t_1}} + \cdots + \frac{l_{t_{m-2}}^2}{w_{t_{m-2}}}\right]^{1/2}\right) \tag{3.9}$$

where $\sum_{\{t_1,\ldots,t_{m-2}\}}$ denotes the sum over possible choices of $t_1 < \cdots < t_{m-2}$ among $1, \ldots, j-1, j+1, \ldots, N-1$. In the limit $L_N \to \infty$ the Kelvin functions in equation (3.9) are exponentially damped and the result reduces to the result of $N-1$ compactified dimensions, as it should.

This completes the results for the topologically generated masses, useful for large values of v^2. The results for special values of n and N are easily read off and need not to be stated explicitly.

Analogous results for small values of v^2 are more difficult to obtain. This is the subject of the following section.

4. Calculation of the effective potential for small values of v^2

As already mentioned, the results (3.3), (3.6) and (3.7) are useful for large values of v due to the exponential decay of the Kelvin functions involved. In order to obtain a power series representation in powers of v^2 we proceed in another direction.

It is reasonable to employ the identity

$$Z_N^{v^2}(s; w_1, \ldots, w_N; u_1, \ldots, u_N) = \left[w_1 u_1^2 + \cdots + w_N u_N^2 + v^2\right]^{-s} + \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(s+j)}{j!\Gamma(s)} Z_N(s+j; w_1, \ldots, w_N; u_1, \ldots, u_N)\, v^{2j} \tag{4.1}$$

where we define

$$Z_N(\nu; w_1, \ldots, w_N; u_1, \ldots, u_N) = \sum_{l_1,\ldots,l_N=-\infty}^{\infty}{}' \left[w_1(l_1-u_1)^2 + \cdots + w_N(l_N-u_N)^2\right]^{-\nu}.$$

Once again using regularization techniques for Mellin transforms, it is easy to see that the only pole of $Z_N(s; w_1, \ldots, w_N; u_1, \ldots, u_N)$ is located at $s = N/2$ with

$$\operatorname{Res} Z_N\left(\tfrac{1}{2}N; w_1, \ldots, w_N; u_1, \ldots, u_N\right) = \frac{\pi^{N/2}}{\sqrt{w_1 \cdots w_N}\,\Gamma(\frac{1}{2}N)}$$

furthermore one has

$$Z_N(-p; w_1, \ldots, w_N; u_1, \ldots, u_N) = -\left[w_1 u_1^2 + \cdots + w_N u_N^2\right]^p$$

for $p \in \mathbb{N}_0$. Employing equation (4.1) in equations (2.7)–(2.10), the expansion of $V_{\text{eff}}(w_i; u_i)$ in powers of v is obtained, where the coefficients are given by properties of the generalized Epstein functions $Z_N(s; w_1, \ldots, w_N; u_1, \ldots, u_N)$. We skip the rather long but essentially routine calculations and give here just the final results for the effective potential, equations (2.7)–(2.10):

(i) $n = 2k$, N even

$$V_T V_{\text{eff}}(w_i; u_i) = -\left(\frac{\sqrt{\pi}}{L_1}\right)^n (-1)^k$$

$$\times \left\{ -\frac{1}{k!}\left[w_1u_1^2+\cdots+w_Nu_N^2+v^2\right]^k \ln(w_1u_1^2+\cdots+w_Nu_N^2+v^2) \right.$$

$$+\sum_{j=0}^{k}\frac{1}{j!(k-j)!}v^{2j}\Big[Z_N\,(j-k;w_1,\ldots,w_N;u_1,\ldots,u_N)$$

$$\times\{\psi(k-j+1)-\psi(k+1)\}+Z_N'\,(j-k;w_1,\ldots,w_N;u_1,\ldots,u_N)\Big]$$

$$+\sum_{j=1,j\neq N/2}^{\infty}(-1)^j\frac{(j-1)!}{(j+k)!}v^{2j+2k}Z_N\,(j;w_1,\ldots,w_N;u_1,\ldots,u_N)$$

$$+(-1)^{N/2}v^{N+2k}\frac{\Gamma(N/2)}{(k+\frac{1}{2}N)!}\Big[PP\ Z_N\left(\tfrac{1}{2}N;w_1,\ldots,w_N;u_1,\ldots,u_N\right)$$

$$\left.+\frac{\pi^{N/2}}{\sqrt{w_1\cdots w_N}\,\Gamma(N/2)}\left\{\psi\left(\frac{N}{2}\right)+\gamma+2\ln\left(\frac{L_1}{2\pi\lambda}\right)\right\}\Big]\right\}. \qquad (4.2)$$

(ii) $n=2k$, N odd

$$V_T V_{\rm eff}(w_i;u_i)=-\left(\frac{\sqrt{\pi}}{L_1}\right)^n(-1)^k$$

$$\times \left\{ -\frac{1}{k!}\left[w_1u_1^2+\cdots+w_Nu_N^2+v^2\right]^k \ln(w_1u_1^2+\cdots+w_Nu_N^2+v^2) \right.$$

$$+\sum_{j=0}^{k}\frac{1}{j!(k-j)!}v^{2j}[Z_N\,(j-k;w_1,\ldots,w_N;u_1,\ldots,u_N)$$

$$\times\{\psi(k-j+1)-\psi(k+1)\}+Z_N'\,(j-k;w_1,\ldots,w_N;u_1,\ldots,u_N)]$$

$$\left.+\sum_{j=1}^{\infty}\frac{(-1)^j(j-1)!}{(j+k)!}v^{2j+2k}Z_N\,(j;w_1,\ldots,w_N;u_1,\ldots,u_N)\right\}. \qquad (4.3)$$

(iii) $n=2k+1$, N even

$$V_T V_{\rm eff}(w_i;u_i)=-\left(\frac{\sqrt{\pi}}{L_1}\right)^n\left\{\Gamma(-k-\tfrac{1}{2})\left[w_1u_1^2+\cdots+w_Nu_N^2+v^2\right]^{k+\frac{1}{2}}\right.$$

$$+\sum_{j=0}^{\infty}(-1)^j\frac{\Gamma\left(j-k-\frac{1}{2}\right)}{j!}v^{2j}$$

$$\left.\times Z_N\left(j-k-\frac{1}{2};w_1,\ldots,w_N;u_1,\ldots,u_N\right)\right\}. \qquad (4.4)$$

(iv) $n=2k+1$, N odd

$$V_T V_{\rm eff}(w_i;u_i)=-\left(\frac{\sqrt{\pi}}{L_1}\right)^n\left\{\Gamma(-k-\tfrac{1}{2})\left[w_1u_1^2+\cdots+w_Nu_N^2+v^2\right]^{k+\frac{1}{2}}\right.$$

$$+\sum_{\substack{j=0\\ j\neq\frac{N+1}{2}+k}}^{\infty}(-1)^j\frac{\Gamma\left(j-k-\frac{1}{2}\right)}{j!}v^{2j}$$

$$\times Z_N\left(j-k-\frac{1}{2}; w_1,\ldots,w_N; u_1,\ldots,u_N\right)$$

$$+(-1)^{\frac{1}{2}(N+1)+k}\frac{\Gamma(N/2)}{(\frac{1}{2}(N+1)+k)!}v^{N+1+2k}$$

$$\times\Bigg[PP\ Z_N\left(N/2; w_1,\ldots,w_N; u_1,\ldots,u_N\right)$$

$$+\frac{\pi^{N/2}}{\sqrt{w_1\cdots w_N}\,\Gamma(N/2)}\left\{\psi\left(\frac{N}{2}\right)+\gamma+2\ln\left(\frac{L_1}{2\pi\lambda}\right)\right\}\Bigg]\Bigg\}. \tag{4.5}$$

In order to determine the topologically generated masses m_{T_i}, we need the series expansion (3.4) of equations (4.2)–(4.5) in powers of the toroidal components A_i of the gauge potential, this is the expansion of $Z_N(s; w_1,\ldots,w_N; u_1,\ldots,u_N)$ in powers of the u_i's. This has been accomplished only recently with the result [35]

$$Z_N\left(s; w_1,\ldots,w_N; u_1,\ldots,u_N\right)$$

$$=\sum_{j=0}^{\infty}\sum_{l=0}^{\left[\frac{j}{2}\right]}\sum_{|\nu|=l}(-1)^{j+l}2^{2l}\frac{1}{(j-2l)!(2\nu)!}\left[w_1u_1^2+\cdots+w_Nu_N^2\right]^{j-2l}x^{2\nu}$$

$$\times\frac{\Gamma(s+j-l)}{\Gamma(s)}\left(\frac{\partial}{\partial w}\right)^{\nu}E_N(s+j-l; w_1,\ldots,w_N) \tag{4.6}$$

where $x=(w_1u_1,\ldots,w_Nu_N)$, $\nu=(\nu_1,\ldots,\nu_N)$

$$\left(\frac{\partial}{\partial w}\right)^{\nu}=\left(\frac{\partial}{\partial w_1}\right)^{\nu_1}\cdots\left(\frac{\partial}{\partial w_N}\right)^{\nu_N}=:\partial_w^{\nu}$$

and we introduced the Epstein zeta function [8, 36–46]

$$E_N\left(\nu; w_1,\ldots,w_N\right)=\sum_{l_1,\ldots,l_N=-\infty}^{\infty}{}'\left[w_1u_1^2+\cdots+w_Nu_N^2\right]^{-\nu}. \tag{4.7}$$

Explicit analytical continuations of equation (4.7) may be found for example in [8]. The properties of $Z_N(s; w_1,\ldots,w_N; u_1,\ldots,u_N)$ relevant for equations (4.2)–(4.5) may be determined with the aid of equation (4.6). Using them in equations (4.2)–(4.5) yields the exact power series of $V_{\text{eff}}(w_i; u_i)$ in powers of M and in powers of the toroidal components of the gauge potential. This is not done explicitly, because the results are very long and we only want to concentrate on the resulting topologically generated masses.

Contributions to the topologically generated masses arise, when we choose $j=1$, $l=0$ or $j=2$, $l=1$ in equation (4.6).

For $n=0,1,2$ zero modes play a crucial role and the limit $v\to 0$ leads to divergent masses $m_{T_j}^2$. In detail the results for arbitrary N are:

(i) $n=0$

$$m_{T_j}^2=-\frac{g^2}{2\pi^2}\frac{L_j^2}{V_T}\left\{-\frac{1}{v^2}+\sum_{l=0}^{\infty}v^{2l}B_l\right\} \tag{4.8}$$

(ii) $n=1$

$$m_{T_j}^2 = -\frac{g^2}{2\pi^{3/2}} \frac{L_j^2}{L_1 V_T} \left\{ -\frac{\sqrt{\pi}}{v} + \sum_{l=0}^{\infty} v^{2l} B_l \right\} \tag{4.9}$$

(iii) $n=2$

$$m_{T_j}^2 = \frac{g^2}{2\pi V_T} \left(\frac{L_j}{L_1}\right)^2 \times \left\{ -\ln v^2 + \left(1+2w_j\partial_{w_j}\right) E'_N(0; w_1, \ldots, w_N) - \sum_{l=1}^{\infty} v^{2l} B_l \right\} \tag{4.10}$$

where we have introduced

$$B_l = (-1)^{l+1} \frac{\Gamma(l+1-\frac{1}{2}n)}{l!} \times \left(1+2w_j\partial_{w_j}\right) PP\, E_N\left(l+1-\tfrac{1}{2}n; w_1, \ldots, w_N\right). \tag{4.11}$$

In the cases where $E_N\left(l+1-\frac{1}{2}n; w_1, \ldots, w_N\right)$ is well defined, the symbol PP is meaningless. For $n > 2$ the limit $v \to 0$ is smooth, and for arbitrary N one finds

(i) $n = 2k$ even, $k \in \mathbb{N}$, $k > 1$

$$m_{T_j}^2 = \frac{g^2}{2\pi^2} \frac{L_j^2}{V_T} \left(\frac{\sqrt{\pi}}{L_1}\right)^n (-1)^{k+1} \left\{ \frac{1}{(k-1)!} v^{n-2} \left[\sum_{l=2}^{k-1} \frac{1}{l} - \ln v^2 \right] + \sum_{l=1}^{k} v^{n-2l} D_l + (-1)^k \sum_{l=k}^{\infty} v^{2l} B_l \right\} \tag{4.12}$$

(ii) $n = 2k+1$ odd, $k \in \mathbb{N}$, $k \geqslant 1$

$$m_{T_j}^2 = -\frac{g^2}{2\pi^2} \frac{L_j^2}{V_T} \left(\frac{\sqrt{\pi}}{L_1}\right)^n \left\{ -\Gamma\left(1-\tfrac{1}{2}n\right) v^{n-2} + \sum_{l=0}^{\infty} v^{2l} B_l \right\} \tag{4.13}$$

with

$$D_l = \frac{1}{(l-1)!(k-l)!} \left(1+2w_j\partial_{w_j}\right) E'_N\left(1-l; w_1, \ldots, w_N\right) \tag{4.14}$$

and B_l given by equation (4.11).

Given these forms of the results it would have been difficult to determine a definite sign for $m_{T_j}^2$. But as we know from section 3, $m_{T_j}^2$ in equations (4.8)–(4.13) is always positive for arbitrary $L_1, \ldots, L_N$. Nevertheless we present these results, because in contrast to equation (3.7) they also contain, in a simple way, the results for $v = 0$ (see section 5).

The results for an equilateral torus are easily recovered by recognizing

$$\partial_{w_j} E_N\left(s; w_1, \ldots, w_N\right)\big|_{w_1=\cdots=w_N=1} = -\frac{s}{N} E_N\left(s; 1, \ldots, 1\right).$$

The corresponding topologically masses are easily obtained from (4.8)–(4.14) and need not be given in detail. Furthermore, the masses for particular values of n and N may easily be stated, so we do not present them here.

Let us remark that it is possible to give different forms of the results by using the reflection formula for the Epstein zeta function [8, 36, 37]

$$\Gamma(s)\pi^{-s}E_N(s;w_1,\dots,w_N) = \frac{1}{\sqrt{w_1\cdots w_N}}\Gamma(\tfrac{1}{2}N-s)\pi^{s-N/2}E_N\left(\frac{N}{2}-s;\frac{1}{w_1},\dots,\frac{1}{w_N}\right). \tag{4.15}$$

For example (4.15) implies

$$\left(1+2w_j\partial_{w_j}\right)E_N(s;w_1,\dots,w_N) = 2\frac{\Gamma(\frac{1}{2}N-s)}{\Gamma(s)}\pi^{2s-N/2}\frac{w_j}{\sqrt{w_1\cdots w_N}}\partial_{w_j}E_N\left(\frac{N}{2}-s;\frac{1}{w_1},\dots,\frac{1}{w_N}\right) \tag{4.16}$$

which may be used in equations (4.8)–(4.14). However, for $v \neq 0$ no essential simplifications are found, but this will change for $v = 0$, where (4.16) enables us to discuss the sign of $m^2_{T_j}$.

5. Generation of imaginary gauge field mass

The Abelian gauge field theory of a massless scalar field coupled to a constant background gauge potential is something special in that the creation of imaginary gauge field mass is possible. For this reason a seperate section is dedicated to this theory.

The starting point is the field equation (see equation (2.1))

$$\left\{-\sum_{i=1}^{N}\left(\frac{\partial}{\partial x_i}-\mathrm{i}A_i\right)^2-\Delta\right\}\phi(\boldsymbol{x},\boldsymbol{y})=0 \tag{5.1}$$

with periodic boundary conditions in the toroidal directions. Going through the calculations of section 2 and 4 one recognizes, that the topologically generated mass is obtained from equations (4.8)–(4.14) just by neglecting the contribution in equation (2.4) resulting from the summation index $l_1 = \cdots = l_N = 0$ in (2.3) and performing the limit $v \to 0$.

Using equation (4.16), for $n > 2$ the result reads

$$m^2_{T_j} = g^2L_1^2\frac{1}{\pi^{\frac{1}{2}(n+N)}L_1^{n+N}}\Gamma\left(\frac{n+N}{2}-1\right)\partial_{w_j}E_N\left(\frac{n+N}{2}-1;\frac{1}{w_1},\dots,\frac{1}{w_N}\right). \tag{5.2}$$

But for $\mathrm{Re}\, s > N/2$ and $w_1 > 0, \dots, w_N > 0$, one finds

$$\partial_{w_j}E_N\left(s;\frac{1}{w_1},\dots,\frac{1}{w_N}\right) = \frac{s}{w_j^2}\sum_{l_1,\dots,l_N=-\infty}^{\infty}{}' l_j^2\left[\frac{l_1^2}{w_1}+\cdots+\frac{l_N^2}{w_N}\right]^{-s-1} > 0 \tag{5.3}$$

and as a consequence $m^2_{T_j} > 0$ for $n > 2$. This was already clear from equations (4.12) and (4.13) because the limit $v \to 0$ is smooth for these cases.

But for $n \leqslant 2$ similar conclusions are not possible. The case $N = 1$ is especially simple in that $E_1(s; w) = 2w^{-s}\zeta_R(2s)$ with the Riemann zeta function ζ_R. Using well known properties of the Riemann zeta function [47, 48], one finds in detail

(i) $n = 0$, $N = 1$

$$m_T^2 = -\frac{g^2}{\pi^2} L\zeta_R(2) < 0 \tag{5.4}$$

(ii) $n = 1$, $N = 1$

$$m_T^2 = -\frac{g^2}{\pi} < 0 \tag{5.5}$$

(iii) $n = 2$, $N = 1$

$$m_T^2 = \frac{g^2}{\pi L}\left\{\tfrac{3}{2} - \ln(2\pi)\right\} < 0. \tag{5.6}$$

So for $n \leqslant 2$, $N = 1$, the topologically generated gauge field mass is always imaginary.

For the remaining dimensions n, N, depending on the compactification lengths $L_1, \ldots, L_N$ involved, the generation of real and imaginary gauge field mass is possible. This will be shown in the rest of this section.

First, equation (4.16) implies for the topologically generated mass

(a) $n = 0$, $N = 2$

$$m_{T_j}^2 = \frac{g^2}{\pi} \partial_{w_j} E'_2\left(0; \frac{1}{w_1}, \frac{1}{w_2}\right) \tag{5.7}$$

(b) $n = 0, 1$, $n + N \geqslant 3$

$$m_{T_j}^2 = \frac{g^2}{\pi^{\frac{1}{2}(N+n)}} L_j^2 L_1^{-N-n} \Gamma\left(\frac{N+n}{2} - 1\right) w_j \partial_{w_j} E_N\left(\frac{N+n}{2} - 1; \frac{1}{w_1}, \ldots, \frac{1}{w_N}\right) \tag{5.8}$$

(c) $n = 2$, $N > 1$

$$m_{T_j}^2 = \frac{g^2}{2\pi V_T}\left(\frac{L_j}{L_1}\right)^2 \left\{\psi\left(\frac{N}{2}\right) - \gamma - 2\ln\pi + 1 \right.$$
$$\left. + \frac{2}{\pi^{N/2}} \Gamma(N/2) \frac{w_j}{\sqrt{w_1 \cdots w_N}} \partial_{w_j} PP\ E_N\left(\frac{N}{2}; \frac{1}{w_1}, \ldots, \frac{1}{w_N}\right)\right\}. \tag{5.9}$$

For these examples, the determination of the sign of $m_{T_j}^2$ is not so simple as in equations (5.2) and (5.3). This time we need the properties of $\partial_{w_j} E_N\left(s; \frac{1}{w_1}, \ldots, \frac{1}{w_N}\right)$ for arguments $s \in [0, N/2]$, where the defining series, equation (5.3), is not convergent and so the positivity is no longer guaranteed. So, in order to make a statement about the sign of $m_{T_j}^2$ an explicit analytic continuation of equation (5.3) to $\mathrm{Re}\, s < N/2$ has to be considered.

Without restriction let us concentrate on $m^2_{T_1}$. Performing as usual a Mellin transformation and re-summations (see for example, [49]), one possible analytic continuation is

$$\begin{aligned}\partial_{w_1} E_N\left(s;\frac{1}{w_1},\ldots,\frac{1}{w_N}\right) =& \frac{2}{\Gamma(s)}\pi^{\frac{1}{2}(N-1)} w_1^{\frac{1}{2}(s-\frac{1}{2}(N+5))}\sqrt{w_2\cdots w_N}\\ &\times\left\{\Gamma(s-\tfrac{1}{2}(N-3))w_1^{\frac{1}{2}(s-\frac{1}{2}(N-3))}\zeta_R(2s+1-N)\right.\\ &+2\pi^{s-\frac{1}{2}(N-3)}\sum_{l_1=1}^{\infty}\sum_{l_2,\ldots,l_N=-\infty}^{\infty}{}' l_1^{\frac{1}{2}(N+1)-s}[w_2l_2^2+\cdots+w_Nl_N^2]^{\frac{1}{2}(s-\frac{1}{2}(N-3))}\\ &\left.\times K_{\frac{1}{2}(N-3)-s}\left(\frac{2\pi l_1}{\sqrt{w_1}}[w_2l_2^2+\cdots+w_Nl_N^2]^{1/2}\right)\right\}. \end{aligned}\tag{5.10}$$

First one recovers the positivity of (5.10) for $\mathrm{Re}\, s > N/2$ already given in equation (5.3), because both terms in the curly brackets are positive. But for $s \in [0, N/2]$ the first term (respectively the finite part of the first term) is negative and because the Kelvin functions are exponentially damped for large arguments, this opens the possibility of imaginary mass generation. However, it may also be shown that the second term in the curly brackets (the sum over the Kelvin functions) is arbitrarily large in a given range of the parameters $w_1, \ldots, w_N$, so that the generation of real gauge field mass is also possible.

To be more concrete, let us exemplify the remarks for example (5.7), the corresponding argument for examples (5.8) and (5.9) is very similar.

First, using (5.10) one finds for (5.7)

$$\begin{aligned} m^2_{T_1} &= 2g^2\sqrt{w_2}\left\{-\tfrac{1}{12}+4w_2^{1/4}\sum_{l_1,l_2=1}^{\infty} l_1^{3/2}l_2^{1/2}K_{1/2}(2\pi l_1 l_2\sqrt{w_2})\right\}\\ &= 2g^2\sqrt{w_2}\left\{-\tfrac{1}{12}+2\sum_{l=1}^{\infty}\frac{l}{\exp(2\pi l\sqrt{w_2})-1}\right\} \end{aligned}\tag{5.11}$$

where in the last equality $K_{1/2}(z) = \sqrt{\pi/2z}\exp(-z)$ [47] has been used.

Obviously the second term in the curly brackets goes to zero (respectively to infinity) if w_2 goes to infinity (respectively to 0). So at some critical value of $w_2 = L_1/L_2$ a transition from real to imaginary mass will take place.

For the examples (5.8) and (5.9) the analogous, but more complicated, results of equation (5.11) may be derived with the conclusion, that in given ranges of the compactification lengths $L_1, \ldots, L_N$, the generated gauge field mass is real or imaginary.

6. Conclusions

In this paper we have considered the quantum field theory of a massive or massless scalar field ϕ, minimally coupled to a constant background Abelian gauge potential. The scalar field ϕ is defined on spacetime $T^N \times \mathbb{R}^n$, so that the toroidal components of the gauge potential become physical parameters. The origin of the topology $T^N \times \mathbb{R}^n$ may, for example, be imagined as imposed externally, the way $S^1 \times \mathbb{R}^n$ is imposed on finite-temperature Euclidean field theories. Another possibility is (see [27] for more details) that a scalar field possessing a self-interaction can generate its own non-trivial spacetime topology through ordinary Casimir energy effects [8].

The effective potential of the theory occupies a central position. Two calculations of the effective potential, convenient in different ranges of the parameters involved, are presented (see (3.3) and (4.2)–(4.5)). The topologically generated gauge field masses $m_{T_j}^2$ of the toroidal components of the gauge field are of particular interest. In contrast to the effective energy, which diverges if the quantum mass M tends to infinity, the gauge field mass is exponentially damped for large M and so vanishes in the limit $M \to \infty$. Furthermore it does not depend on a renormalization scale and we have shown, that for a massive scalar field the masses m_{T_j} are real for arbitrary mass M of the quantum field and for arbitrary compactification lengths $L_1, \ldots, L_N$ of the torus. As a result $A_i = 0$ is a local minimum of the vacuum energy. So for a massive scalar field, the calculations show a possible mechanism for the quantum generation of a gauge boson mass analogous to the electric screening mass in finite-temperature field theory.

However, for a massless scalar field we have seen that $A_i = 0$ may also be a local maximum of the vacuum enery. For $n = 0, 1, 2$ there always exist given ranges of compactification lengths, such that an imaginary gauge field mass is generated, which reveals a quantum mechanism for gauge symmetry break down. The new phase of the gauge theory generates masses for scalar fields, by a mechanism which is almost a perfect inversion of the standard Higgs mechanism for generating gauge field mass by giving scalar fields a vacuum expectation value (for a detailed discussion see [27]).

In comparison for $n > 2$ the limit $M \to 0$ is smooth and imaginary gauge field mass generation is impossible.

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